ON STEADY FLOW OF A CONDUCTING FLUID IN A RECTANGULAR TUBE WITH TWO NONCONDUCTING WALLS, AND TWO CONDUCTING ONES PARALLEL TO AN EXTERNAL MAGNETIC FIELD

(OB USTANOVIVSHEMSIA TECHENII PROVODIASHCHEI ZHIDKOSTI V PRIAMOUGOL'NOI TRUBE S DVUMIA NEPROVODIASHCHIMI STENKAMI I DVUMIA PROVODIASHCHIMI, PARALLEL'NYMI VNESHNEMU MAGNITNOMU POLIU)

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1. As was pointed out by Shercliff [1], if the external magnetic field $\mathbf{H}^\circ$ is uniform, and the velocity field and induced electric and magnetic fields do not depend on the coordinate z measured along the axis of the tube, there exists a solution of the equations of steady motion of a conducting viscous incompressible fluid along a tube such that

$$\mathbf{v} = v\mathbf{i}_z, \qquad \mathbf{H} = \mathbf{H}^\circ + H_z\mathbf{i}_z \tag{1.1}$$

where

$$\mathbf{E} = \frac{c}{4\pi\sigma}\,\mathrm{grad}\, H_z \times \mathbf{i}_z + \frac{\mu}{c}\,\mathbf{H} \times \mathbf{v}, \qquad \mathbf{j} = \frac{c}{4\pi}\,\mathrm{grad}\, H_z \times \mathbf{i}_z \tag{1.2}$$

Here $\mathbf{E}$ and $\mathbf{j}$ are the electric field and current density in the fluid, σ its conductivity and μ its magnetic permeability.

We choose the x-axis in the direction of the field H°. Assuming that $-\partial p/\partial z = P =$ const, and supposing that external volume forces are absent, we obtain for H_z and v the following equations:

$$\triangle H_z + \frac{4\pi\mu\sigma H^\circ}{c^2}\frac{\partial v}{\partial x} = 0 \tag{1.3}$$

$$\triangle v + \frac{H^\circ\mu}{4\pi\eta}\frac{\partial H_z}{\partial x} = -\frac{P}{\eta} \qquad \left(\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1.4}$$

Here η is the coefficient of viscosity of the fluid.

The boundary conditions for H_z on the walls (contour) S of the tube (assuming them fixed) are written as:

on the nonconducting parts of the wall

$$\partial H_z / \partial S = 0 \quad \text{on } S \tag{1.5}$$

on the ideally conducting parts (where n is the normal to the wall)

$$\partial H_z / \partial n = 0 \quad \text{on } S \tag{1.6}$$

To these is clearly added the condition

$$v = 0 \quad \text{on } S \tag{1.7}$$

The case of nonconducting walls was considered in [1], and the case of ideally conducting walls in [2]. In both cases the solution is represented by trigonometric series obtained by use of the method of particular solutions. It is possible in the same way to solve the problem of a rectangular tube whose walls perpendicular to the external magnetic field are ideally conducting, and those parallel to it are nonconducting*. It appears much more difficult to consider the question for a tube with ideally conducting walls parallel to H° and nonconducting ones perpendicular to H°. Since an exact solution of this problem does not, to the best of our knowledge, exist in the literature at the present time, we give here some considerations related to finding such a solution and its investigation. In particular, the problem is reduced to an integral equation of the first kind, which is easily solved by numerical means for small and moderate values of the Hartmann number, and admits of asymptotic investigation in the case when this number is large.

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* It should be noted that such a form of the solution appears to be very disadvantageous in the case of large values of the Hartmann number, because the convergence of the series deteriorates rapidly as that number increases. The situation here is analogous to that occurring in the theory of the diffraction of waves by a body of finite dimensions, where the convergence of the series obtained by the method of particular solutions rapidly deteriorates with increase of the ratio of a characteristic dimension of the body to the wavelength (for a cylinder, sphere, etc.). In the present case, as in that of diffraction, one may attempt to find a practically useful solution by summing the resulting badly convergent series, turning it into a complex integral and then transforming this by one means or another to obtain a new form of the solution that is useful even for large values of the Hartmann number.

2. We proceed to the solution of the stated problem. Choosing the x, y-axes as shown in Fig. 1, setting $OA = l = 2a$, and $OB = d$, and denoting by I the total current flowing in through the ideally conducting wall OA and flowing out through BC (per unit axial length of the tube), we obtain the following boundary conditions for the field H_z:

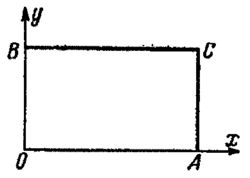

Fig. 1.

$$H_z = 2\pi I / c \quad \text{at} \quad x = 0, \quad H_z = -2\pi I / c \quad \text{at} \quad x = l \tag{2.1}$$

$$\partial H_z / \partial y = 0 \quad \text{at} \quad y = 0, \quad \partial H_z / \partial y = 0 \quad \text{at} \quad y = d \tag{2.2}$$

Furthermore

$$v|_{x=0} = 0, \quad v|_{x=l} = 0, \quad v|_{y=0} = 0, \quad v|_{y=d} = 0 \tag{2.3}$$

Introducing the functions

$$u = \frac{1}{2\gamma}\left[\frac{H^\circ\mu}{4\pi\eta} H_z + \frac{P}{\eta}(x-a)\right], \qquad \gamma = \frac{\mu H^\circ}{2c}\sqrt{\frac{\sigma}{\eta}} \tag{2.4}$$

and setting

$$\alpha = \frac{1}{4\gamma\eta}\left(Pl - \frac{H^\circ\mu I}{c}\right) \tag{2.5}$$

we obtain from (1.3) and (1.4) the equations

$$\triangle u + 2\gamma\frac{\partial v}{\partial x} = 0, \qquad \triangle v + 2\gamma\frac{\partial u}{\partial x} = 0 \tag{2.6}$$

with the boundary conditions on the walls OB and AC

$$u|_{x=0} = -\alpha, \quad u|_{x=l} = \alpha; \quad v|_{x=0} = 0, \quad v|_{x=l} = 0 \tag{2.7}$$

For the combinations $p = u + v$ and $q = u - v$ we find

$$\Delta p + 2\gamma\,\partial p / \partial x = 0, \qquad \Delta q - 2\gamma\partial q / \partial x = 0 \tag{2.8}$$

$$p|_{x=0} = -\alpha, \quad q|_{x=0} = -\alpha, \quad p|_{x=l} = \alpha, \quad q|_{x=l} = \alpha \tag{2.9}$$

We set finally

$$p = e^{-\gamma(x-a)} S, \qquad q = e^{\gamma(x-a)}\, t \tag{2.10}$$

and introduce the unknown stream function

$$\left.\frac{\partial v}{\partial y}\right|_{y=0} = f_0(x), \qquad \left.\frac{\partial v}{\partial y}\right|_{y=d} = f_d(x) \qquad (0 \leqslant x \leqslant l) \tag{2.11}$$

Then for S and t we obtain the equations

$$\Delta S - \gamma^2 S = 0, \qquad \Delta t - \gamma^2 t = 0 \tag{2.12}$$

with boundary conditions

$$S\big|_{x=0} = -\alpha e^{-\gamma a}, \qquad S\big|_{x=l} = \alpha e^{\gamma a}, \qquad \frac{\partial S}{\partial y}\bigg|_{y=0} = e^{\gamma(x-a)} f_0(x)$$
$$\frac{\partial S}{\partial y}\bigg|_{y=d} = e^{\gamma(x-a)} f_d(x) \tag{2.13}$$

$$t\big|_{x=0} = -\alpha e^{\gamma a}, \qquad t\big|_{x=l} = \alpha e^{-\gamma a}, \qquad \frac{\partial t}{\partial y}\bigg|_{y=0} = -e^{-\gamma(x-a)} f_0(x)$$
$$\frac{\partial t}{\partial y}\bigg|_{y=d} = -e^{-\gamma(x-a)} f_d(x) \tag{2.14}$$

For the determination of the functions S and t under the boundary conditions (2.13) and (2.14) one may use the Green's function $G(\xi, \eta, x, y)$ for the rectangular region under consideration, given by the equation

$$2\pi G(\xi, \eta, x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \{K_0[\gamma \sqrt{(x_m-\xi)^2 + (y_n-\eta)^2}] +$$
$$+ K_0[\gamma \sqrt{(x_m-\xi)^2 + (y_n'-\eta)^2}] - K_0[\gamma \sqrt{(x_m'-\xi)^2 + (y_n-\eta)^2}] -$$
$$- K_0[\gamma \sqrt{(x_m'-\xi)^2 + (y_n'-\eta)^2}]\} \tag{2.15}$$

where $K_0(z)$ is the Macdonald function, and*

$$x_m = 2ml + x, \quad x_m' = 2ml - x \quad (m = 0, \pm 1, \pm 2, \ldots)$$
$$y_n = 2nd + y, \quad y_n' = 2nd - y \quad (n = 0, \pm 1, \pm 2 \ldots). \tag{2.16}$$

and satisfying Equation (2.12) and boundary conditions of the form**

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* We note that

$$x'_{-m} = -x_m, \quad x'_{-(m-1)} - l = -(x_m - l), \quad y'_{-n} = -y_n, \quad y'_{-(n-1)} - d = -(y_n - d).$$

** It is easily obtained by the method of images from the basic solution $K_0(\gamma R)$ of Equation (2.12), where $R = \sqrt{[(x-\xi)^2 + (y-\eta)^2]}$ is the distance from the fixed point (ξ, η) to the variable point (x, y). We note that for small γl and γd Expression (2.15) for the function $G(\xi, \eta, x, y)$ is advantageously transformed for practical use into another form on which we will not dwell here.

$$G(0,\eta,x,y)=0,\qquad G(l,\eta,x,y)=0,\quad \left.\frac{\partial G}{\partial\eta}\right|_{\eta=0}=0,\quad \left.\frac{\partial G}{\partial\eta}\right|_{\eta=d}=0 \quad (2.17)$$

By use of Green's formula we obtain the following expressions:

$$S(x,y)=\int_0^l \{G(\xi,d,x,y)\,f_d(\xi)-G(\xi,0,x,y)\,f_0(\xi)\}\,e^{\gamma(\xi-a)}\,d\xi-$$
$$-\alpha\int_0^d \{G_\xi'(l,\eta,x,y)\,e^{\gamma a}+G_\xi'(0,\eta,x,y)\,e^{-\gamma a}\}\,d\eta \qquad (2.18)$$

Consequently

$$p(x,y)=e^{-\gamma(x-a)}S(x,y)=$$
$$=\int_0^l \{G(\xi,d,x,y)\,f_d(\xi)-G(\xi,0,x,y)\,f_0(\xi)\}\,e^{\gamma(\xi-x)}\,d\xi-$$
$$-\alpha\int_0^d \{G_\xi'(l,\eta,x,y)\,e^{\gamma(l-x)}+G_\xi'(0,\eta,x,y)\,e^{-\gamma x}\}\,d\eta$$
$$q(x,y)=e^{\gamma(x-a)}\,t(x,y)= \qquad (2.19)$$
$$=-\int_0^l \{G(\xi,d,x,y)\,f_d(\xi)-G(\xi,0,x,y)\,f_0(\xi)\}\,e^{-\gamma(\xi-x)}\,d\xi-$$
$$-\alpha\int_0^d \{G_\xi'(l,\eta,x,y)\,e^{-\gamma(l-x)}+G_\xi'(0,\eta,x,y)\,e^{\gamma x}\}\,d\eta$$

Thus

$$v(x,y)=\tfrac{1}{2}(p-q)= \qquad (2.20)$$
$$=\int_0^l \{G(\xi,d,x,y)\,f_d(\xi)-G(\xi,0,x,y)\,f_0(\xi)\}\cosh\gamma(x-\xi)\,d\xi-$$
$$-\alpha\left\{\sinh\gamma(l-x)\int_0^d G_\xi'(l,\eta,x,y)\,d\eta-\sinh\gamma x\int_0^d G_\xi'(0,\eta,x,y,)\,d\eta\right\}$$
$$u(x,y)=\tfrac{1}{2}(p+q)= \qquad (2.21)$$
$$=\int_0^l \{G(\xi,d,x,y)\,f_d(\xi)-G(\xi,0,x,y)\,f_0(\xi)\}\sinh\gamma(\xi-x)\,d\xi-$$
$$-\alpha\left\{\cosh\gamma(l-x)\int_0^d G_\xi'(l,\eta,x,y)\,d\eta+\cosh\gamma x\int_0^d G_\xi'(0,\eta,x,y)\,d\eta\right\}$$

The last equation can be greatly simplified, because the integrals with respect to η appearing in its right-hand side do not, as will appear immediately, depend upon y and d, and are thus functions only of x. Consequently we obtain according to (2.15), for example

$$\int_0^d G_\xi'(0,\eta,x,y)\,d\eta = \frac{\gamma}{2\pi}\sum_{m=-\infty}^{\infty}\Big\{x_m\sum_{n=-\infty}^{\infty}\Big[\int_0^d \frac{K_1(\gamma\sqrt{x_m^2+(y_n-\eta)^2})}{\sqrt{x_m^2+(y_n-\eta)^2}}\,d\eta +$$
$$+\int_0^d \frac{K_1(\gamma\sqrt{x_m^2+(y_{-n}'-\eta)^2})}{\sqrt{x_m^2+(y_{-n}'-\eta)^2}}\,d\eta\Big] - x_m'\sum_{n=-\infty}^{\infty}\Big[\int_0^d \frac{K_1(\gamma\sqrt{x_m'^2+(y_n-\eta)^2})}{\sqrt{x_m'^2+(y_n-\eta)^2}}\,d\eta +$$
$$+\int_0^d \frac{K_1(\gamma\sqrt{x_m'^2+(y_{-n}'-\eta)^2})}{\sqrt{x_m'^2+(y_{-n}'-\eta)^2}}\,d\eta\Big]\Big\} \tag{2.22}$$

Setting in the first integral on the right $y_n - \eta = 2nd + y - \eta = t$, and in the second $y_{-n}' - \eta = -(2nd + y + \eta) = -t$ and combining the resulting integrals with respect to t, we find

$$\sum_{n=-\infty}^{\infty}\Big[\int_0^d \frac{K_1(\gamma\sqrt{x_m^2+(y_n-\eta)^2})}{\sqrt{x_m^2+(y_n-\eta)^2}}\,d\eta + \int_0^d \frac{K_1(\gamma\sqrt{x_m^2+(y_{-n}'-\eta)^2})}{\sqrt{x_m^2+(y_{-n}'-\eta)^2}}\,d\eta\Big] =$$
$$= \sum_{n=-\infty}^{\infty}\int_{2nd+y-d}^{2nd+y+d}\frac{K_1(\gamma\sqrt{x_m^2+t^2})}{\sqrt{x_m^2+t^2}}\,dt = \int_{-\infty}^{\infty}\frac{K_1(\gamma\sqrt{x_m^2+t^2})}{\sqrt{x_m^2+t^2}}\,dt = \frac{\pi e^{-\gamma|x_m|}}{\gamma|x_m|} \tag{2.23}$$

Here we have used the known formula

$$\int_0^\infty \frac{K_1(\gamma\sqrt{a^2+t^2})\,dt}{\sqrt{a^2+t^2}} = \frac{\pi e^{-\gamma|a|}}{2\gamma|a|} \tag{2.24}$$

valid for arbitrary real values of $a \gtrless 0$.

Substituting into (2.22) the result obtained and its analog corresponding to the replacement of x_m by x_m' in Equation (2.23), we arrive at the relation

$$\int_0^d G_\xi'(0,\eta,x,y)\,d\eta = \frac{1}{2}\sum_{m=-\infty}^{\infty}\Big\{\frac{x_m}{|x_m|}e^{-\gamma|x_m|} - \frac{x_m'}{|x_m'|}e^{-\gamma|x_m'|}\Big\} =$$
$$= \sum_{m=-\infty}^{\infty}\frac{x_m}{|x_m|}e^{-\gamma|x_m|} = \frac{\sinh\gamma(l-x)}{\sinh\gamma l} \tag{2.25}$$

where it is taken into account that $x_m' = -x_m = -(2ml + x)$. Analogously we find

$$\int_0^d G_\xi'(l,\eta,x,y)\,d\eta = -\frac{\sinh\gamma x}{\sinh\gamma l} \tag{2.26}$$

Substitution of the results obtained into (2.20) and (2.21) gives

$$v(x, y) = \int_0^l \{G(\xi, d, x, y) f_d(\xi) - G(\xi, 0, x, y) f_0(\xi)\} \cosh \gamma (x - \xi) d\xi + \\ + \frac{2\alpha \sinh\gamma x \sinh\gamma(l - x)}{\sinh \gamma l} \tag{2.27}$$

$$u(x, y) = \int_0^l \{G(\xi, d, x, y) f_d(\xi) - G(\xi, 0, x, y) f_0(x)\} \sinh \gamma (\xi - x) d\xi - \\ - \alpha \frac{\sinh\gamma(l - 2x)}{\sinh \gamma l} \tag{2.28}$$

The condition $v(x, 0) = v(x, d) = 0$ is now reduced to the form

$$\int_0^l \{G(\xi, d, x, 0) f_d(\xi) - G(\xi, 0, x, 0) f_0(\xi)\} \cosh \gamma (x - \xi) d\xi = \\ = -2\alpha \frac{\sinh\gamma(l - x) \sinh\gamma x}{\sinh \gamma l} \tag{2.29}$$

$$\int_0^l \{G(\xi, d, x, d) f_d(\xi) - G(\xi, 0, x, d) f_0(\xi)\} \cosh \gamma (x - \xi) d\xi = \\ = -2\alpha \frac{\sinh\gamma(l - x) \sinh\gamma x}{\sinh \gamma l} \tag{2.30}$$

From (2.15) it follows immediately that

$$G(\xi, d, x, d) = G(\xi, 0, x, 0), \qquad G(\xi, d, x, 0) = G(\xi, 0, x, d)$$

Therefore, subtracting (2.29) from (2.30), we obtain

$$\int_0^l \{G(\xi, 0, x, 0) - G(\xi, 0, x, d)\} [f_d(\xi) + f_0(\xi)] \cosh \gamma (x - \xi) d\xi = 0 \\ (0 < x < l) \tag{2.31}$$

Thus it must be true that*

$$f_d(\xi) = -f_0(\xi) \tag{2.32}$$

and Equations (2.29) and (2.30) are reduced to one, namely

$$\int_0^l \{G(\xi, d, x, 0) + G(\xi, 0, x, 0)\} f_0(\xi) \cosh \gamma (x - \xi) d\xi = 2\alpha \frac{\sinh\gamma x \sinh\gamma(l - x)}{\sinh\gamma l} \tag{2.33}$$

* This follows immediately from the symmetry character of the solution being sought.

This is an integral equation for the function $f_0(\xi)$. If the function $f_0(\xi)$ is determined from this equation, then from it by means of Equations (2.27), (2.28) and (2.32) u and v can be determined, that is, the complete solution of the problem be found.

For this, Equations (2.27) and (2.28) acquire the form

$$u = \int_0^l [G(\xi, d, x, y) + G(\xi, 0, x, y)] f_0(\xi) \sinh \gamma (x - \xi)\, d\xi - \alpha \frac{\sinh \gamma (l - 2x)}{\sinh \gamma l} \quad (2.34)$$

$$v = -\int_0^l [G(\xi, d, x, y) + G(\xi, 0, x, y)] f_0(\xi) \cosh \gamma (x - \xi)\, d\xi + 2\alpha \frac{\sinh \gamma x \sinh \gamma (l - x)}{\sinh \gamma l} \quad (2.35)$$

If $[\partial v/\partial y]_{y=0} = 0$, only the terms outside the integrals remain in the right-hand sides of Equations (2.34) and (2.35), and these equations are the (known) exact solution of the one-dimensional problem for Equations (2.6) and (2.7), obtained under the assumption that the entire process depends upon x and not upon y. In other words, these terms give the exact solution of the problem of the corresponding flow of fluid not in a tube of rectangular cross-section but between two parallel plates of infinite extent in the direction of the y-axis.

We note also that for $a = 0$ the right-hand side of Equation (2.33) vanishes, so that $f_0(\xi) = 0$, and consequently $u = v = 0$ according to (2.34) and (2.35). From Equation (2.4) it then follows that

$$H_z = \frac{4\pi P}{H^\circ \mu}(a - x) \quad (2.36)$$

and from the condition $a = 0$ it follows that $P/H^\circ\mu = I/2ac$, so that

$$H_z = \frac{2\pi I}{c}\left(1 - \frac{x}{a}\right) \quad (2.37)$$

Hence for the current density $\mathbf{j}$ we obtain the expression

$$\mathbf{j} = \frac{c}{4\pi} \operatorname{grad} H_z \times \mathbf{i}_z = -\frac{I}{l}\mathbf{i}_y \quad (2.38)$$

that is, the current flows parallel to the y-axis, with its density constant in the x-direction. Thus we have a static regime, the fluid being at rest, so that the variation of pressure is balanced by the electromagnetic forces exerted on the current $\mathbf{j}$ by the field $\mathbf{H}^\circ$.

From Equation (2.33) it is evident that

$$f_0(\xi) = \alpha \Phi(\xi) \quad (2.39)$$

where the function $\Phi(\xi)$ does not depend upon a.

Equations of an analogous type are obtained, according to (2.34) and

(2.35), for v and u. For $\Phi(\xi)$ we obtain the equation

$$\int_0^l \{G(\xi, 0, x, 0) + G(\xi, d, x, 0)\}\,\Phi(\xi)\cosh\gamma(x-\xi)\,d\xi = \frac{2\sinh\gamma x\sinh\gamma(l-x)}{\sinh\gamma l} \quad (2.40)$$

From Expression (2.15) for $G(\xi, \eta, x, y)$ it is evident, in particular, that for $\gamma d \gg 1$ the term $G(\xi, d, x, 0)$ in the left-hand side of Equation (2.40) is very small in comparison with $G(\xi, 0, x, 0)$.

Since this term reflects the effect of finite dimension of the tube section in the direction of the y-axis upon the distribution of the function $f_0(\xi) = [\partial v/\partial y]_{y=0}$ along the wall $y = 0$, that is, the effect upon this distribution of the wall at $y = d$, it follows from what has been said that for $\gamma d \gg 1$ this influence can be neglected, and with greater accuracy the greater is the parameter γd. For $d = \infty$, which corresponds to the case of a section in the form of a half-strip rather than a rectangle, it disappears completely. Equation (2.40) then takes the form

$$\int_0^l g(\xi, 0, x, 0)\,\varphi(\xi)\cosh\gamma(x-\xi)\,d\xi = \frac{2\sinh\gamma x\sinh\gamma(l-x)}{\sinh\gamma l} \quad (2.41)$$

where $\phi(\xi)$ is the corresponding solution and $g(\xi, 0, x, 0)$ the Green's function for the half-strip, obtained from (2.15) by passing to the limiting case $d = \infty$, and equal to

$$g(\xi, \eta, x, y) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\{K_0(\gamma\sqrt{(x_m-\xi)^2+(y-\eta)^2}) + K_0(\gamma\sqrt{(x_m-\xi)^2+(y+\eta)^2}) - K_0(\gamma\sqrt{(x_m'-\xi)^2+(y-\eta)^2}) - K_0(\gamma\sqrt{(x_m'-\xi)^2+(y+\eta)^2})\} \quad (2.42)$$

since in (2.15) all terms with $n \neq 0$ vanish.

Introducing the dimensionless variables $z = x/l$, $\zeta = \xi/l$ and setting $\gamma l = M$ and $l\phi(zl) = \psi(z)$, we obtain finally*

$$\int_0^l g(l\zeta, 0, lz, 0)\,\psi(\zeta)\cosh M(z-\zeta)\,d\zeta = \frac{2\sinh Mz\sinh M(1-z)}{\sinh M} \quad (2.43)$$

where

* There is clearly no danger of confusing this $z = x/l$ with the coordinate z in the direction of the axis of the tube.

$$g\,(l\zeta, 0, lz, 0) = \sum_{m=-\infty}^{\infty} [K_0\,(M\,|\,2m + z - \zeta\,|) - K_0\,(M\,|\,2m - z - \zeta\,|)] \quad (2.44)$$

This is a one-parameter equation for the unknown function $\psi(\zeta)$, M being the Hartmann number.

In what follows, certain consequences resulting from this equation are considered.

3. For small and moderate values of M, Equation (2.43) can be solved numerically, reducing it to a system of coupled linear equations, where one must take account of the fact* that $\psi(0) = \psi(1) = 0$. For large values of its argument the function $K_0(u)$ is asymptotically equal to $e^{-u}\sqrt{(\pi/2u)}$; consequently in the case of large values of M the series (2.44) for the function $g(l\zeta, 0, lz, 0)$ converges very rapidly. For sufficiently large M, Equation (2.44) may be limited to three terms, namely (3.1)

$$g\,(l\zeta, 0, lz, 0) \approx \frac{1}{\pi}\,\{K_0\,(M\,|\,z - \zeta\,|) - K_0\,[M\,(z + \zeta)] - K_0\,[M\,(2 - z - \zeta)]\}$$

according to which Equation (2.43) takes the following form:

$$\frac{1}{\pi}\int_0^1 \{K_0\,(M\,|\,z - \zeta\,|) - K_0\,(M\,|\,z + \zeta\,|) - K_0\,[M\,(2 - z - \zeta)]\}\,\psi\,(\zeta) \times$$
$$\times \cosh M\,(z - \zeta)\,d\zeta = \frac{2\sinh Mz\sinh M\,(1 - z)}{\sinh M} \quad (3.2)$$

In numerical solution of this equation account must be taken of the fact that its kernel becomes infinite at $\zeta = z$. We divide the full interval of integration (0, 1) into sufficiently small subintervals (ζ_i, ζ_{i+1}) and take advantage of the approximate equality

$$\int_{\zeta_i}^{\zeta_{i+1}} g\,(l\zeta, 0, lz, 0)\cosh M\,(z - \zeta)\,\psi\,(\zeta)\,d\zeta \approx \psi\,(\zeta_i^\circ) \times$$
$$\times \int_{\zeta_i}^{\zeta_{i+1}} g\,(l\zeta, 0, lz, 0)\cosh M\,|\,z - \zeta\,|\,d\zeta \quad (3.3)$$

where ζ_i° is some average intermediate value of ζ, for example

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* Because $f_0(\xi) = [\partial v/\partial y]_{y=0}$ vanishes at $\xi = 0$ and $\xi = 1$, the velocity v is equal to zero on these walls.

$\zeta_i^\circ = 1/2(\zeta_i + \zeta_{i+1})$. Here it is essential that the integral appearing in this formula can be found as an indefinite one in terms of known functions, if use is made of the formula*

$$\int e^{\pm t} K_0(t)\,dt = te^{\pm t}[K_0(t) \pm K_1(t)] \tag{3.4}$$

and taking into account Expression (3.1) for $g(l\zeta, 0, lz, 0)$ and the fact that**

$$\cosh M(z-\zeta) = \tfrac{1}{2}[e^{M(z-\zeta)} + e^{-M(z-\zeta)}]$$

4. We consider the asymptotic form of Equation (2.43) obtained for $M \to \infty$. In so doing we proceed from the fact that for $M \to \infty$ over almost all of the interval of integration the asymptotic equality is valid

$$K_0(M|z-\xi|)\cosh M(z-\xi) \approx \frac{1}{2}\sqrt{\frac{\pi}{2M|z-\zeta|}} \tag{4.1}$$

and analogously

$$K_0[M(z+\zeta)]\cosh M(z-\zeta) \approx \frac{1}{2}\sqrt{\frac{\pi}{2M(z+\zeta)}}\begin{cases} e^{-2M\zeta} & \text{for } z>\zeta \\ e^{-2Mz} & \text{for } z<\zeta \end{cases} \tag{4.2}$$

$$K_0[M(2-z-\zeta)]\cosh M(z-\zeta) \approx \frac{1}{2}\sqrt{\frac{\pi}{2M(2-z-\zeta)}}\begin{cases} e^{-2M(1-\zeta)} & \text{for } z<\zeta \\ e^{-2M(1-z)} & \text{for } z>\zeta \end{cases} \tag{4.3}$$

where (4.2) is valid under the condition $M(z+\zeta) \gg 1$ and (4.3) under the condition $M|2-(z+\zeta)| \gg 1$.

Equation (3.2) can be put into the form

$$\frac{1}{2\sqrt{2\pi M}}\left\{\int_0^1 \frac{\psi(\zeta)\,d\zeta}{\sqrt{|z-\zeta|}} - e^{-2Mz}\int_z^1 \frac{\psi(\zeta)\,d\zeta}{\sqrt{z+\zeta}} - e^{-2M(1-z)}\int_0^z \frac{\psi(\zeta)\,d\zeta}{\sqrt{2-z-\zeta}} - \right.$$
$$\left. - \int_0^z \frac{\psi(\zeta)\,e^{-2M\zeta}}{\sqrt{z+\zeta}}\,d\zeta - \int_z^1 \frac{\psi(\zeta)\,e^{-2M(1-\zeta)}}{\sqrt{2-z-\zeta}}\,d\zeta\right\} = \frac{2\sinh Mz\sinh M(1-z)}{\sinh M} - \delta(z) \tag{4.4}$$

or, if we put $\psi(\zeta) = 2\sqrt{(2\pi M)}\,\chi(\zeta)$, in the form

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* This formula, easily verified by differentiation, is a special case of a very general one given by us in [3] and repeated in [4], p. 696.

** The integral in Equation (3.3) obviously reduces simply to the sum of integrals of the form (3.4) with constant multipliers.

$$\int_0^1 \frac{\chi(\zeta)\, d\zeta}{\sqrt{|z-\zeta|}} = \frac{2\sinh Mz \sinh M(1-z)}{\sinh M} - \delta(z) + R(z) \tag{4.5}$$

Here

$$\delta(z) = \int_0^1 \left[K_0(M|z-\zeta|)\cosh M(z-\zeta) - \frac{1}{2}\sqrt{\frac{\pi}{2M|z-\zeta|}}\right]\psi(\zeta)\, d\zeta -$$

$$-\int_0^z \left\{K_0[M(z+\zeta)]\cosh M(z-\zeta) - \frac{1}{2}\sqrt{\frac{\pi}{2M(z+\zeta)}}\, e^{-2M\zeta}\right\}\psi(\zeta)\, d\zeta -$$

$$-\int_z^1 \left\{K_0[M(z+\zeta)]\cosh M(z-\zeta) - \frac{1}{2}\sqrt{\frac{\pi}{2M(z+\zeta)}}\, e^{-2Mz}\right\}\psi(\zeta)\, d\zeta -$$

$$-\int_0^z \left\{K_0[M(2-z-\zeta)]\cosh M(z-\zeta) - \frac{1}{2}\sqrt{\frac{\pi}{2M(2-z-\zeta)}}\, e^{-2M(1-z)}\right\}\psi(\zeta) d\zeta -$$

$$-\int_z^1 \left\{K_0[M(2-z-\zeta)]\cosh M(z-\zeta) - \frac{1}{2}\sqrt{\frac{\pi}{2M(2-z-\zeta)}}\, e^{-2M(1-\zeta)}\right\}\psi(\zeta) d\zeta \tag{4.6}$$

$$R(z) = e^{-2Mz}\int_z^1 \frac{\chi(\zeta)\, d\zeta}{\sqrt{z+\zeta}} + e^{-2M(1-z)}\int_0^z \frac{\chi(\zeta)\, d\zeta}{\sqrt{2-z-\zeta}} \tag{4.7}$$

On the right-hand side of Equation (4.4) the first term can be written as

$$\frac{2\sinh Mz \sinh M(1-z)}{\sinh M} = 1 - \frac{e^{-2Mz} + e^{-2M(1-z)} - e^{-2M}}{1 - e^{-2M}}$$

that is, for very large M it is practically equal to unity over almost the entire interval $(0 \leqslant z \leqslant 1)$, but vanishes at its ends. As for $\delta(z)$, it vanishes in the whole interval as M increases, because the integrands in each of the integrals appearing in it are significantly different from zero only within a region whose width is of the order $O(M^{-1})$ in the vicinity of the point z.

Thus, for example, at a point z sufficiently removed from the edges (0, 1) of the interval, where the function $\psi(\zeta)$ changes relatively little within a width of order $O(M^{-1})$, we have the approximation

$$\int_0^1 \left\{ K_0(M|z-\zeta|)\cosh M(z-\zeta) - \frac{1}{2}\sqrt{\frac{\pi}{2M|z-\zeta|}} \right\} \psi(\zeta) d\zeta \approx$$

$$\approx \frac{2\psi(z)}{M}\int_0^\infty \left[K_0(t)\cosh t - \frac{1}{2}\sqrt{\frac{\pi}{2t}} \right] dt \tag{4.8}$$

Hence it is evident that this integral has in any case the order $O(M^{-1})$. However, the integral on the right is equal to zero, which is easily found using Equation (3.4), which gives

$$\int_0^\infty \left[K_0(t)\, e^t - \sqrt{\frac{\pi}{2t}} \right] dt = -1, \qquad \int_0^\infty K_0(t)\, e^{-t}\, dt = 1$$

or

$$\int_0^\infty \left[K_0(t) \cosh t - \frac{1}{2} \sqrt{\frac{\pi}{2t}} \right] dt = 0$$

From this it is evident that in regions where the function $\psi(\zeta)$ changes sufficiently smoothly, the integral under consideration will have a very high degree of smallness.

The remaining integrals appearing in $\delta(z)$ can be estimated in analogous fashion.

Therefore in seeking an asymptotic solution we will in the first approximation disregard the value of $\delta(z)$. Similarly we neglect also the terms

$$\int_1^z \frac{\psi(\zeta)\, e^{-2M\zeta}\, d\zeta}{\sqrt{z+\zeta}}, \qquad \int_z^1 \frac{\psi(\zeta)\, e^{-2M(1-\zeta)}\, d\zeta}{\sqrt{2-z-\zeta}} = \int_0^z \frac{\psi(\zeta_1)\, e^{-2M\zeta_1}\, d\zeta_1}{\sqrt{z_1+\zeta_1}} \quad (z_1 = 1 - z_1)$$

where account is taken of the fact that $\psi(1-\zeta) = \psi(\zeta)$. As for the second and third terms in curly brackets on the left in Equation (4.4), they must be retained, because although the factors e^{-2Mz} and $e^{-2M(1-z)}$ appearing in them also decrease extremely rapidly with distance from the end points $z = 0$ and $z = 1$ of the interval, they just bring about the vanishing of the left-hand side of Equation (4.4) at $z = 0$ and $z = 1$, as occurs in the right-hand side of the same equation with $\delta(z)$ neglected; and without them the right- and left-hand sides of the equation could not be equal to each other at the end points.

Equation (4.5) now acquires (approximately) the form

$$\int_0^1 \frac{\chi(\zeta)\, d\zeta}{\sqrt{|z-\zeta|}} = \frac{2 \sinh Mz \sinh M(1-z)}{\sinh M} + e^{-} \int_z^1 \frac{\chi(\zeta)\, d\zeta}{\sqrt{z+\zeta}} + e^{-2M(1-z)} \int_0^z \frac{\chi(\zeta)\, d\zeta}{\sqrt{2-z-\zeta}} \qquad (4.9)$$

If the right-hand side of Equation (4.5) or (4.9) is known, then $\chi(z)$ can be found in quadratures, using the known solution of the singular integral equation of the form

$$\int_0^1 \frac{\chi(\zeta)\, d\zeta}{\sqrt{|z-\zeta|}} = w(z) \qquad (0 \leqslant z \leqslant 1) \qquad (4.10)$$

where the function $w(z)$ is regarded as known [5,6].

We use henceforth the form of the solution of this equation given in [6], namely

$$\chi(z) = -\frac{1}{\sqrt{2\pi}\,\Gamma^2\left(\frac{3}{4}\right)} \frac{1}{z^{1/4}} \frac{d}{dz} \int_z^1 \frac{\sqrt{\zeta}\, d\zeta}{(\zeta - z)^{1/4}} \frac{d}{d\zeta} \int_0^\zeta \frac{w(\sigma)\, d\sigma}{\sigma^{1/4}(\zeta - \sigma)^{1/4}} \qquad (4.11)$$

Note. Substituting into (4.11) for $w(z)$ the right-hand side of Equation (4.5), we obtain a new (exact) form of the integral equation for the unknown function $\chi(z)$, and substituting the right-hand side of Equation (4.9) we obtain an approximate integral equation for $\chi(z)$.

In [6], Equation (4.11) is given in a rather different form, because the equation to be solved is written in the form

$$\int_0^a \frac{\psi(y)\, dy}{|x^2 - y^2|^p} = \varphi(x), \qquad 0 \leqslant x \leqslant a \qquad (4.12)$$

where $a =$ const, $p =$ const, $0 < p < 1$.

Formula (4.10) is obtained from the solution of Equation (4.12) given in [6] with $p = 1/2$ and corresponding change of variables.

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